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BARGAINING THEORY FOR GAMES WITH TRANSFERABLE VALUE*

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ABSTRACT

This paper presents an existence proof of a bargaining equilibrium set B_*, in the case of games with transferable value, by making use of the Knaster Kuratowski Mazurkiewicz (KKM) Theorem. As a corollary proof of existence of the usual bargaining set B_1 is obtained. Whereas previous proofs of B_1 existence have made use of fixed point arguments, use of the KKM theorem provides an insight into possible extensions of the existence proof to the nontransferable value case.

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1. INTRODUCTION

The standard proof of nonemptiness of the bargaining set $B_1(M)$, for a coalition M in a voting game with transferable value, makes use of a fixed point argument (Peleg, 1967b). In this paper a proof of nonemptiness of $B_1(M)$ is obtained by showing the nonemptiness of two subsets of $B_1(M)$, namely the kernel, K(M), and a new bargaining set $B_*(M)$.

Moreover the method of proof presented here uses the Knaster, Kuratowski, Mazurkiewicz (KKM) Theorem (1929) rather than a fixed point argument. This theorem gives sufficient conditions for the nonemptiness of the intersection of a finite family of subsets of a simplex. In the situation examined here, the sets under consideration are equilibrium sets, associated with the members of a coalition, M, in the pareto set V(M) for the coalition.

Since the KKM theorem can be generalized to the case of a family of subsets of an arbitrary compact convex subset of a topological vector space (see Fan, 1961, and Schofield, 1983b for discussion) this suggests that the proof procedure presented here can be extended to obtain existence theorems for equilibrium bargaining sets in the more general case of general characteristic function games without transferable value.

2. THE BARGAINING SETS AND KERNEL

Here we give a brief formal definition of the bargaining set and kernel.

A game with transferable value (t.v.) for a society $N = \{1,\dots,n\}$ is a function $v : 2^N \to \mathbb{R}$ where 2^N is the power set of N (i.e., the class of all subsets, or coalitions, of N).

The number, v(M), is the value associated with coalition M. We assume for any individual i in N that $v(\{i\}) \geq 0$.

Of particular interest is a simple t.v. game defined in terms of a class, $\mathbb{D}$, of winning coalitions, such that

$$v(M) > 0 \text{ iff } M \in \mathbb{D}$$

and

$$v(M) = 0 \text{ iff } M \notin \mathbb{D} .$$

A simple weighted majority game (with t.v.) is a simple t.v. game where winning coalitions are defined in the following way. Each player $\{1,\dots,n\}$ is assigned a weight $w(i)$. The weight of coalition M is $w(M) = \sum_{i \in M} w(i)$. A number q, with $q > 1/2\, w(N)$, is specified such that $M \in \mathbb{D}$ iff $w(M) \geq q$. In such a game we also assume $v(M) = 1$ wherever $M \in \mathbb{D}$. A simple weighted majority game is often written

$$[q : q(1),\dots,w(n)].$$

A simple q-majority game is a simple weighted majority game where each player has weight 1. Thus the game may be written $[q : 1,\dots,1]$. A simple majority game is a simple q-majority game with q being the smallest integer which is strictly greater than n/2.

For a general transferable value game, v, and coalition M let V(M) be the subset of $\mathbb{R}^n$ defined as follows:

$$x \in V(M) \text{ iff } \quad \text{(i) } x_j = 0 \text{ for all } j \notin M.$$
$$\text{(ii) } x_i \geq 0 \text{ for all } i \in M.$$
$$\text{(iii) } \sum_{i \in M} x_i = v(M).$$

A <u>payoff configuration</u> (p.c.) is a pair (x,M) where M is a coalition and $x = (x_1, \ldots, x_n)$ belongs to v(M).

One payoff configuration (y,C) <u>dominates</u> another (x,M) iff $y_i > x_i$ for each $i \in C$. In this case write (y,C)dom(x,M). The <u>core</u> is the set of payoff configurations which are undominated. For a typical (proper simple) voting game the core will be empty. See Schofield (1980) for a general argument to this effect, which is applicable to both the transferable and nontransferable cases. However, the core will be nonempty if there exists a <u>veto player</u>; a veto player is a player (or party) that belongs to every winning coalition. When the core is empty, every payoff configuration will be unstable (i.e., dominated by another). We therefore look for a solution theory to select those payoff configurations that might be "less unstable" in some sense.

Consider a p.c., (x,M), dominated by another, (y,C). The latter may be considered a "threat" by any player i in $C \cap M$ to a player j in $M \backslash C$. On the other hand suppose that there exists a p.c., (x,D), dominating (y,C), where $j \in D$, such that $z_j \geq x_j$. Then the threat by i to j may be countered by j without loss.

We make this more formal.

If L,J are two subsets of N, let T_{LJ} be the family of supersets of L which do not intersect J. Thus

$$T_{LJ} = \{A \subset N : L \subset A \text{ and } J \cap A = \Phi\}$$

<u>Definition 1</u>

Let (x,M) be a p.c. and L,J two disjoint subsets of the coalition M.

(a) An <u>objection</u> by L against J with respect to (x,M) is a p.c., (y,C) such that

(i) $C \in T_{LJ}$

(ii) $y_i > x_i$ for all $i \in L$

(iii) $y_i \geq x_i$ for all $i \in C$.

(b) A <u>counter objection</u> by J against L's objection, (y,C), is a p.c., (x,D), such that

(i) $D \in T_{JL}$

(ii) $z_j \geq x_j$ for all $j \in J$

(iii) $z_j \geq y_j$ for all $j \in D$.

(c) An objection (y,C) by L again J is said to be <u>justified</u> if there is no counter objection by J to (y,C). If L has a justified objection against J with respect to the p.c., (x,M), then write LP(x)J.

Definition 2

(a) A p.c., (x,M), is called B_1-stable if to any objection by an individual i against and individual $j \in M\backslash\{i\}$, there is a counter objection by j. Let $B_1(M)$ be the set of B_1-stable payoff vectors for M, and call $B_1(M)$ the B_1-bargaining set for M. Thus $B_1(M) = \{x \in V(M) : (x,M) \text{ is } B_1\text{-stable}\}$

(b) A p.c., (x,M), is called B_2-stable if to any objection by an individual i against any subgroup $J \subset M\backslash\{i\}$, there is a counter objection by J. Let $B_2(M)$ be the set of B_2-stable payoff vector for M. Thus $B_2(M) = \{x \in V(M) : (x,M) \text{ is } B_2\text{-stable}\}$.

Suppose we write $iP_2(x)j$ when individual i has a justified objection against a subgroup $J \subset M\backslash\{i\}$ which contains j. Clearly $iP(x)j$ implies that $iP_2(x)j$.

Moreover,

$$B_1(M) = \{x \in V(M) : iP(x)j \text{ for no } i,j \in M\}$$

and

$$B_2(M) = \{x \in V(M) : iP_2(x)j \text{ for no } i,j \in M\}$$

and so $B_2(M) \subset B_1(M)$.

Note that if i objects with (y,C) then this may be regarded as an objection against the subgroup $M\backslash C$. For B_1-stability each individual j in $M\backslash C$ must be able to counter object by $(z(j),D(j))$, say. For B_2-stability the whole group $M\backslash C$ must be able to form a counter objection (z,D) such that $M\backslash C \subset D$ and $z_j \geq x_j$ for all $j \in M\backslash C$.

Clearly a B_2 counter objection will be more difficult to effect than a B_1 counter objection.

We now introduce a third solution notion, the kernel which belongs to $B_1(M)$.

Definition 3

Let $V : 2^N \to \mathbb{R}$ be a transferable value game.

(i) Let (x,M) be a p.c. For any other coalition C, define the excess of C over M at x to be

$$e_x(C) = v(C) - x(C)$$

where

$$x(C) = \sum_{i \in C} x_i.$$

(This excess is the amount the members of C stand to gain over their current payoff if they can form this coalition.) Suppose now that i,j are two players in the coalition M.

(ii) Define the surplus of i over j at $x \in V(M)$ to be $S_{ij}(x) = \max_C \{e_x(C) : C \in T_{ij}\}$. Thus i's surplus over j is the maximum excess of i over x, across all coalitions that include i but exclude j.

(iii) Say that i outweighs j (with respect to (x,M)) iff

(i) $x_j > 0$

(ii) $S_{ij}(x) > S_{ji}(x)$,

and in this case write $iQ(x)j$.

(iv) Say the p.c., (x,M) is K-stable if for no $i,j \in M$ is it the case that $iQ(x)j$. Then the kernel, $K(M)$, for M is the set $K(M) = \{x \in V(M) : (x,M) \text{ is K-stable}\}$.

The above definitions can be made more general by referring to a coalition structure, S. A coalition structure (cs), S, is a disjoint partition $(M_1,\ldots,M_r)$ of N. A payoff configuration (x,S) is a pair (x,S) where S is a coalition structure and for each M in S, $\sum_{i \in M} x_i = v(M)$. A p.c. (x,S) belongs to the Bargaining Set $B_1(S)$ iff for each M in S it is the case that (x,M) belongs to $B_1(M)$, in the sense of definition 1. All definitions and proofs of existence for $B_1(M)$ and $K(M)$, etc., extend to the bargaining set and kernel $B_1(S)$ and $K(S)$ for coalition structures.

For notational convenience we shall only work with $B_1(M)$ and $K(M)$ etc.

3. PROPERTIES OF THE BARGAINING SETS

In this section we examine the properties of $K(M)$, $B_1(M)$ and $B_2(M)$ and introduce a new bargaining set $B_*(M)$. First of all we show, for a game with transferable value, that the kernel for a coalition belongs to the B_1-bargaining set of that coalition.

Lemma 1 (a generalization of Davis and Maschler, 1967, lemma 3.1). Let v be a game with transferable value. Let M be a coalition, and (x,M) a p.c. Let L,J be two disjoint subsets of M. If L has an objection (y,C) against J w.r.t. (x,M) and there is some coalition

$D \in T_{JL}$ such that $e_x(D) \geq e_x(C)$, then J has a counter objection against L's objection, (y,C).

Proof: Since (y,C) is an objection,

$$\begin{aligned} e(C) &= v(C) - x(C) \\ &= v(C) - y(C) + \sum_C (y_i - x_i) > 0. \end{aligned}$$

Define the counter objection (z,D) by

$$z_i = x_i \quad \text{for } i \in D - C - J$$

$$z_i = y_i \quad \text{for } i \in D \cap C$$

Consider $\{z_i : i \in J\}$ such that

$$\sum_J z_i = v(D) - \sum \{z_j : j \in D - J\}.$$

Therefore

$$\sum_J (z_i - x_i) = v(D) - \sum_{D \cap C} y_i - \sum_{D-C-J} x_j - \sum_J x_i.$$

Now

$$\sum_{D \cap C} y_j = v(C) - \sum_{C-D} y_i \leq v(C) - \sum_{C-D} x_j.$$

Hence

$$\sum_J (z_i - x_i) \geq v(D) - v(C) + \sum_{C-D} x_i - \sum_{D-C} x_i.$$

Thus

$$\sum_J (z_i - x_i) \geq [v(D) - \sum_{D-C} x_i - \sum_{D \cap C} x_i]$$

$$- [v(C) - \sum_{C-D} x_i - \sum_{D\cap C} x_i]$$

$$= e_x(D) - e_x(C).$$

If $e_x(D) \geq e_x(C)$ then it is possible to find a p.c. (z,D) such that $z_i \geq x_i$ for each $i \in J$. Thus J has a counter objection against (y,C).

Q.E.D.

<u>Lemma 2</u>: Let v be a game with transferable value, and let M be a coalition. Then $K(M)$ is a subset of $B_1(M)$.

<u>Proof</u>: Clearly if $M \notin \mathbb{D}$, so $v(M) = 0$ then $K(M) = B_1(M)$ assigns the payoff 0 to each player in M.

Suppose therefore that $v(M) > 0$, and let (x,M) be a p.c. such that $x \notin B_1(M)$. Then there exist two players i,j in M such that i has a justified objection (y,C) against j. Clearly if $x_j = 0$ then j could form a counter objection $(y,\{j\})$ where y satisfies $y_j = 0$. So assume $x_j > 0$. Suppose there exists a coalition $D \in T_{ji}$ such that $e_x(D) \geq e_x(C)$. By lemma 1, j would have a counter objection to (y,C). Consequently $e_x(C) > e_x(D)$ for all $D \in T_{ji}$. Thus $S_{ij}(x) \geq e_x(C) > S_{ji}(x) \geq e_x(D)$ and so $iQ(x)j$. Thus $x \notin K(M)$. Hence $K(M)$ is a subset of $B_1(M)$.

Q.E.D.

Although the kernel has a number of attractive features, and can be fairly readily computed (see Aumann, Peleg and Rabinowitz, 1965), it

can contain counter-intuitive points, as the following example illustrates.

<u>Example 1</u>

Consider the simple weighted majority voting game

$$\left[\begin{array}{lllllll} & 1 & 2 & 3 & 4 & 5 & 6 \\ 8: & 1, & 1, & 2, & 3, & 3, & 5 \end{array}\right]$$

where $v(M) = 1$ for each coalition M with $w(M) \geq 8$. Let coalition $M = \{1,3,6\}$, and consider $x \in V(M)$. Player 1 may object to $\{3,6\}$ and form a winning coalition $\{1,2,4,5\}$. Consequently $e_x(\{1,2,4,5\}) = 1 - x_1 = S_{13} = S_{16}$. In the same way player 3 may object to $\{1,6\}$ forming $\{3,4,5\}$ so $e_x(\{3,4,5\}) = 1 - x_3 = S_{31} = S_{36}$, while player 6 may object to $\{1,3\}$ forming $\{5,6\}$, so $e_x(\{5,6\}) = 1 - x_6 = S_{61} = S_{63}$. In other words each of the three players $\{1\}$, $\{3\}$, $\{6\}$ pivots in the sense of being able to form a winning coalition with players outside M. As a result the requirement that $S_{13} = S_{31}$ implies that $x_1 = x_3$, while $S_{16} = S_{61}$ implies that $x_1 = x_6$. Thus for $x \in K_1(\{1,3,6\})$ it is necessary that $x_1 = x_3 = x_6 = 1/3$.

More generally it can be shown in a simple weighted majority voting game if two players i,j in a coalition M have weights $w(i)$, $w(j)$ such that $w(i) \geq w(j)$, then if $x \in K(M)$ it is necessary that $x_i \geq x_j$ (see Schofield, 1982). However, as the example above shows, it is not necessarily that case that $w(i) > w(j)$ implies that $x_i > x_j$. In fact, Example 1 corresponds to a cabinet coalition situation that occurred in Denmark in 1957. In that case cabinet posts were distributed to the three parties in the ratio $x_1 : x_3 : x_6 = 0.19 : 0.25 : 0.56$. Since this is quite different from the kernel prediction, it is reasonable to infer that the kernel does not model bargaining capabilities in a very plausible fashion. Some empirical justification for this comment can be found in an earlier analysis of European coalition government portfolio distributions (Schofield, 1976).

In a simple q-majority game, it is easy to show that for a winning coalition M, $x \in K(M)$ iff $x_i = x_j$ for each pair i,j in M. However, in such a game the bargaining set $B_1(M)$ may contain highly inequitable and counter-intuitive payoffs, as the following example illustrates.

Example 2

Let v be the simple majority game with twenty-five players and q = 13. Suppose each winning coalition has value 1.

Let $$x = (1/7,\ldots,1/7, 0 \cdots 0)$$

be a payoff vector associated with the coalition M = {1,...,13} which pays 1/7 to the first seven members of M, and 0 to the remaining six. Suppose player 8 objects to player 1 with coalition C = {8,...,20}, paying each of these 1/13. For 1 to counter object he needs seven members of C, and may pay them 7/13 altogether, leaving 6/13 > 1/7 for himself. Indeed it is easy to show that any objection by a player in {8,...,13} to a player in {1,...,7} has a counter. Consequently the vector x is B_1-stable. However, the vector x is not B_2-stable. To see this let player 8 object to the group J = {1,...,7}, giving each player outside J the payoff 1/18. Clearly J has no counter objection.

More generally, B_2-stability requires that the payoffs in a simple q-majority game are equitable, as the following result (Schofield, 1978) indicates.

Lemma 3

Let v be a simple q-majority game, M a winning coalition and $x \in V(M)$.

(a) Suppose (i) that there exist two individuals i,j in M such that $x_i < x_j$ and (ii) there is a group J in M of size at most (n − q) such that $x(J) = \sum_{k \in J} x_k \geq \frac{n-q}{q}$ then x does not belong to $B_2(M)$.

(b) If $x_i = \frac{1}{|M|}$ for all $i \in M$, then $x \in B_2(M)$.

Although this result shows that B_2 is nonempty for each coalition in a simple q-majority game, in a general transferable value game, $B_2(M)$ may be empty for some coalition. For this reason we introduce a solution notion B_*, which is a subset of B_1, and can be shown to be nonempty for each coalition.

First of all we mention a general procedure for constructing solution theories. Suppose coalition M and some value v(M) is given. Suppose further that for each $x \in V(M)$ there is defined a power relation R(x), which is a subset of M x M. For each i in M let

$E_R(i) = \{x \varepsilon V(M) : jR(x)i$ for no $j \varepsilon M\}$ be the i^{th} equilibrium set. Then $B_R(M) = \bigcap_{i\varepsilon M} E_R(i)$ is the bargaining set in V(M), associated with the power relation R.

Now consider the power relation P(x), of definition 1, restricted to M x M. For individual $i \varepsilon M$, define

$$E_1(i) = E_P(i) = \{x \varepsilon V(M) : jP(x)i \text{ for no } j \varepsilon M\}.$$

Then clearly the bargaining set $B_1(M)$ which we defined above is given by

$$B_1(M) = \bigcap_{i\varepsilon M} E_1(i).$$

In precisely the same way the solution sets B_2 and K for coalition M can be defined from the equilibrium sets associated with the power relations P_2 and Q respectively.

<u>Definition 4</u>

Let $v : 2^N \to \mathbb{R}$ be a transferable value game. Let M be a coalition and i,j be two individuals in M.

(a) For $x \varepsilon V(M)$ define $jP_*(x)i$ iff the following are both satisfied:

(i) $x_i > 0$

(ii) for some $I \varepsilon T_{ij}$, $jP(x)I$

and there is no $J \varepsilon T_{ji}$ with $I \cap J \neq \Phi$ such that $iP(x)J$ (here P(x) is the power relation of definition 1).

(b) Let $E_*(i) = \text{closure}[E_{P_*}(i)]$ where $E_{P_*}(i)$ is the i^{th} equilibrium set $\{x \varepsilon V(M) : jP_*(x)i$ for no $j \varepsilon M\}$.

(c) Define the B_*-bargaining set for M to be $B_*(M) = \bigcap_{i\varepsilon M} E_*(i)$.

The idea behind this definition is that even when j has a justified objection against a subgroup I, say, containing i, then i may block this objection if it can find a justified objection against some subgroup J which contains j, where J has nonempty intersection with I. Note that it is more difficult to make a P-justified objection than a P_*-justified objection, and so $B_*(M)$ is a subset of $B_1(M)$. We present this more formally.

<u>Lemma 4</u>: Let v be a transferable value game, and let M be any coalition. Then

$$B_2(M) \subset B_*(M) \subset B_1(M)$$

<u>Proof</u>: (i) Suppose first of all that $v(M) > 0$.

(a) To show $B_*(M) \subset B_1(M)$.

Suppose that for some $i,j \varepsilon M$, $x \varepsilon V(M)$ it is the case that $jP(x)i$, even though not $(jP_*(x)i)$. Note first of all that $x_i > 0$, otherwise i has a counter objection $(y,\{i\})$, where $y_i = 0$. Since i must be able to block j's objection, there must exist $J \varepsilon T_{ji}$ such that $iP(x)J$ and $\{i\} \cap J \neq \Phi$. But this implies $i \varepsilon J$, which contradicts $J \varepsilon T_{ji}$. Hence, for all $i,j \varepsilon M$, $jP(x)i$ implies $jP_*(x)i$. By definition, the set $\{x \varepsilon V(M) : iP(x)j\}$ is an open set in V(M). As a consequence, $E_P(i) = \{x \varepsilon V(M) : jP(x)i$ for no $i \varepsilon M\}$ is closed. Moreover $E_{P_*}(i) \subset E_P(i)$, for each $i \varepsilon M$. To see this suppose that

$x \in V(M)\setminus E_P(i)$. Then there exists some j such that $jP(x)i$. But by the above this implies $jP_*(x)i$. Hence $x \in V(M)\setminus E_{P_*}(x)$. Since $E_P(i)$ is closed, the closure $E_*(i)$ of $E_{P_*}(i)$ also belongs to $E_P(i)$. But then

$$B_*(M) = \bigcap_{i\in M} E_*(i) \subset \bigcap_{i\in M} E_P(i) = B_1(M).$$

(b) Suppose now that $x \in E_{P_2}(i)$ for some $i \in M$. By definition there exists no $j \in M\setminus\{i\}$ and no $I \subset M\setminus\{j\}$ such that $jP(x)I$. Consequently for no $j \in M$ is it the case that $jP_*(x)i$. Thus $x \in E_{P_*}(i)$. Clearly $E_{P_2}(i) \subset E_*(i)$ for each $i \in M$, and so

$$B_2(M) \subset B_*(M).$$

(ii) In the case that $v(M) = 0$ then clearly any vector x in $V(M)$, satisfying $x_i = 0$ for all i in M, belongs to $B_2(M)$, $B_*(M)$ and $B_1(M)$. Thus all three sets are identical.

Q.E.D.

The standard proof for the existence of a solution theory $B_R(x)$ depends on the assumption that, for each $x \in V(M)$, the power relation, $R(x)$, is "acyclic." In the next section we shall present the existence theorem for B_* and B_1. First of all we show that $P_*(x)$ is acyclic.

<u>Definition 5</u>

A relation $R \subset M \times M$ is

(i) <u>asymmetric</u> iff aRb implies $\text{not}(bRa)$ for any $a,b \in M$

(ii) <u>acyclic</u> iff for any subset $\{a_1,\dots,a_t\}$ of M it is the case that $a_1Ra_2 \cdots Ra_t$ implies $\text{not}(a_tRa_1)$.

(iii) <u>cyclic</u> iff R is not acyclic.

<u>Lemma 5</u> (a generalization of Davis and Maschler, 1967, theorem 3.1) Let v be a transferable value game, and M any coalition. Then for each $x \in V(M)$ the relations $P_*(x) \subset M \times M$, $Q(x) \subset M \times M$ and $P(x) \subset M \times M$ are all acyclic.

<u>Proof</u>

(i) Suppose that $P_*(x)$ is cyclic on $M \times M$, for some $x \in V(M)$. Write R for $P_*(x)$ and take M to be $\{1,\dots,t,\dots\}$ such that $1R\,2R \dots Rt\,R1$.

For each pair $(r, r+1)$ let C_r be the coalition that maximizes the excess $e(C)$ where C contains r but not $(r+1)$. Let C_s be the coalition that maximizes

$$e(C_r),\ r = 1,\dots,t.$$

Consider $(s-1)\,R\,s\,R(s+1)$ and suppose that $(s-1) \notin C_s$. Let $(s-1)$ object to $M - C_{s-1}$.

If $M - C_{s-1}$ belongs to C_s then, by Lemma 3 and the fact that $e(C_s) \geq e(C_{s-1})$, player s has a counter objection, with C_s, against $(s-1)$. But this means that $(s-1)$ does not have a justified objection against a subgroup containing s. By contradiction $M - C_{s-1} \not\subset C_s$ and so $(M - C_{s-1}) \cap (M - C_s) \neq \Phi$.

However, $(s - 1) P (M - C_{s-1})$ and $sP(M - C_s)$, which contradicts $(s - 1) R S$. Consequently $(s - 1) \varepsilon C_s$.

By induction $1 \varepsilon C_s$.

By cyclicity $t R 1$ and so $t \varepsilon C_s$, and by induction again $(s + 1) \varepsilon C_s$.

But C_s is the coalition which s uses to form an objection against $(s + 1)$. By definition therefore $(s + 1) \not\varepsilon C_s$. By the contradiction $P_*(x)$ must be acyclic.

(ii) The proof for the acyclicity of $Q(x)$ proceeds in the same way. For some $x \varepsilon V(M)$ let $Q = Q(x)$ and assume there is a Q cycle $1Q\ 2Q \cdots Qt\ Q1$. Write $t + 1 = 1$. For each pair $(r,r + 1)$, for $r = 1,\ldots,t$, let C_r be the coalition that maximizes the excess $\{e(C) : C \varepsilon T_{r,r+1}\}$. Let C_s be the coalition that maximizes $e(C_r) : r = 1,\ldots,t$. Now consider $(s - 1)Q\ s\ Q(s + 1)$ and suppose that $(s - 1) \not\varepsilon C_s$. By definition $C_s \varepsilon T_{s,s+1}$ so $s \varepsilon C_s$. If $(s - 1) \not\varepsilon C_s$ then $C_s \varepsilon T_{s,s-1}$. But by the assumption $e(C_s) \geq e(C_{s-1})$ where C_{s-1} is the coalition that maximizes $\{e(C) : C \varepsilon T_{s-1,s}\}$. But we have assumed that $(s - 1)Qs$ and so

$$e(C_{s-1}) > e(C) \quad \text{for all } C \varepsilon T_{s,s-1}.$$

By contradiction $(s - 1) \varepsilon C_s$. By the induction procedure, individual 1 belongs to C_s. Moreover, $t\ Q\ 1$ and $1 \varepsilon C_s$ implies that $t \varepsilon C_s$. By induction again $(s + 1) \varepsilon C_s$. By assumption C_s is the coalition that

maximizes $\{e(C) : C \varepsilon T_{s,s+1}\}$ and so $(s + 1) \not\varepsilon C_s$. By this contradiction the cycle $1Q\ 2Q \cdots Qt\ Q1$ may not exist. Thus for all $x \varepsilon V(M)$, the relation $Q(x)$ on $M \times M$ is acyclic.

(iii) Lemma 4 has established, for each $i,j \varepsilon M$ and any $x \varepsilon V(M)$ that $jP(x)i$ implies $jP_*(x)i$. Moreover, Lemma 2 shows that if $jP(x)i$ then $S_{ji}(x) > S_{ij}(x)$ or $jQ(x)i$. Thus if for some $x \varepsilon V(M)$, $P(x)$ is cyclic, then both $P_*(x)$ and $Q(x)$ are cyclic, contradicting the first two parts of this lemma. Thus $P(x)$ is acyclic.

Q.E.D.

The importance of acyclicity of a power relation $R(x)$ on $M \times M$ is that there must exist some individual i, say, who is maximal, or at the "top of the pecking order," in the sense that, for no $j \varepsilon M$, is it the case that $jR(x)i$. There must also exist an individual, k, at the "bottom" i.e., $kR(x)j$ for no $j \varepsilon M$. The standard proof of this is by induction, using the finiteness of M. A second property that results from acyclicity of $R(x)$, for all x, is that the family of equilibrium sets $\{E_R(i) : i \varepsilon M\}$ is a cover for $V(M)$.

<u>Lemma 6</u>: If for each $x \varepsilon V(M)$ the relation $R(x)$ is acyclic, then

$$\bigcup_{i \varepsilon M} E_R(i) = V(M).$$

<u>Proof</u>: Suppose that $x \varepsilon V(M)$ such that $x \varepsilon E_R(i)$ for no $i \varepsilon M$. Then for <u>every</u> $i \varepsilon M$, there exists some $j \varepsilon M$ such that $jR(x)i$. But this

violates acyclicity.

Q.E.D.

Having established the properties of power relations, we turn, in the next section, to the existence theorem for bargaining sets.

4. AN EXISTENCE THEOREM FOR BARGAINING SETS

The original proof of nonemptiness of the bargaining set $B_1(M)$, for each coalition M, relied on a fixed point argument (Peleg, 1967b, 1969; Billera, 1970) making use of the acyclicity of the underlying power relation (Davis and Maschler, 1967). Here we shall show existence of the B_*-bargaining set by making use of a proof procedure developed by Border (1983) which relies on the Knaster-Kuratowski-Mazurkiewicz (1929) or KKM Theorem. To present the KKM Theorem we let Δ represent the standard $(m-1)$-dimensional simplex in $\mathbb{R}^m$. The i^{th} vertex of Δ is $e^i = (0,\ldots,1,\ldots,0)$ with 1 in the i^{th} position. Let $M = \{1,\ldots,m\}$. For any set $X = \{x_1,\ldots,x_j\}$ of points in Δ let con(X) be the convex hull of X.

<u>KKM Theorem</u>: Let $\{F_1,\ldots,F_m\}$ be a family of closed subsets of Δ, indexed by M, such that, for each subset A of M, the convex hull, $\text{con}\{e^i : i \in A\}$, belongs to $\bigcup_{i\in A} F_i$. Then $\bigcap_{i=1}^{m} F_i$ is nonempty (and compact).

To make use of this theorem in the context of a proof of the equilibrium set $B_R(M)$ for a power relation R, we note first that the

set V(M) in $\mathbb{R}^n$ may essentially be identified with the $(m-1)$ dimensional simplex.

A <u>power relation</u> R we regard as a correspondence $R : \Delta \to M \times M$, where for each $x \in \Delta$, R(x) is a relation on M, and thus a subset of M x M. We shall say that R is <u>acyclic</u> iff, for each $x \in \Delta$, R(x) is acyclic. A power relation R is <u>open</u> iff, for each $i,j \in M$ the set

$$\Delta_{ji} = \{x \in \Delta : jR(x)i\} \text{ is open in } \Delta.$$

As before the i^{th} equilibrium set of R is

$$E_R(i) = \{x \in \Delta : jR(x)i \text{ for no } j \in M\}.$$

Define the equilibrium set of R on M to be

$$B_R(M) = \bigcap_{i\in M} E_R(i).$$

Note that if R is an open power relation, then

$$E_R(i) = \bigcap_{j\in M} (\Delta\backslash\Delta_{ji})$$

is the intersection of closed sets, and therefore closed.

Now define $\sum_i = \{x \in \Delta : x_i = 0\}$.

If we consider the power relation P(x) of definition 1, restricted to M x M, and regard x as a member of Δ rather than V(M) then clearly, for any vector $x \in \Delta$ such that $x_i = 0$, it is the case that jP(x)i for no $j \varepsilon M$. Consequently $\sum_i \subset E_R(i)$. This property is important in the proof of existence of the equilibrium set.

Theorem 1 (Peleg, 1967a; Border, 1983).

Let R be a power relation on Δ which is acyclic and open. Suppose further that for each $i \varepsilon M$, $\sum_i \subset E_R(i)$. Then $B_R(M) = \bigcap_{i \varepsilon M} E_R(i)$ is nonempty.

Proof: For each $i \varepsilon M$, define

$$F_R(i) = \{x \varepsilon \Delta : iR(x)j \text{ for no } j \varepsilon M\}.$$

Pick any subset A of M, and let $x \varepsilon \text{con}\{e^i : i \varepsilon A\}$. Since A is a finite set, and $R(x)$ is an acyclic relation on A, there exists some integer $k \varepsilon A$, such that $kR(x)j$ for no $j \varepsilon A$. Moreover, suppose $i \notin A$. Since $x \varepsilon \text{con}\{e^j : j \varepsilon A\}$, it is the case that $x_i = 0$ and so $x \varepsilon \sum_i$ or $x \varepsilon E_R(i)$. But then $jR(x)i$ for no $j \varepsilon M$. In particular $kR(x)i$ for no $i \varepsilon M\backslash A$. Thus $x \varepsilon F_R(k)$. But $k \varepsilon A$ and so

$$\text{con}\{e^i : i \varepsilon A\} \subset \bigcup_{i \varepsilon A} F_R(i).$$

Moreover for each $k \varepsilon M$,

$$F_R(k) = \bigcap_{j \varepsilon M} (\Delta \backslash \Delta_{kj})$$

is the intersection of closed sets and therefore closed.
By the KKM Theorem, $\bigcap_{k \varepsilon M} F_R(k) \neq \Phi$.

But $B_R(M) = \bigcap_{i \varepsilon M} E_R(i) = \bigcap_{i \varepsilon M} \bigcap_{j \varepsilon M} (\Delta \backslash \Delta_{ij}) = \bigcap_{k \varepsilon M} F_R(k)$.

Thus $B_R(M) \neq \Phi$.

Q.E.D.

Theorem 2 (Billera, 1970).

Let $M = \{1,\ldots,m\}$ and let $\{F_i : i \varepsilon M\}$ be a family of closed subsets of the simplex Δ in $\mathbb{R}^m$, such that for each $i \varepsilon M$, $\sum_i \subset F_i$. If $\bigcup_{i \varepsilon M} F_i = \Delta$, then $\bigcap_{i \varepsilon M} F_i \neq \Phi$.

Proof: Define the power relation R as follows:
For each $x \varepsilon \Delta$, $iR(x)j$ iff $d(x,F_i) > d(x,F_j)$ and $x_j > 0$. Here $d : \Delta \times \Delta \rightarrow \mathbb{R}$ is any distance function on Δ, so for any subset F of Δ, $d(x,F)$ is the distance $\inf_{y \varepsilon F} d(x,y)$, while $x = (x_1,\ldots,x_m)$ is the usual coordinate system for $\mathbb{R}^m$. Now the power relation R is open and acyclic on Δ. Moreover, if $x_i = 0$ then $jR(x)i$ for no $j \varepsilon M$. By Theorem 1, the set

$$B_R(M) = \{x \varepsilon \Delta : iR(x)j \text{ for no } i,j \varepsilon M\}$$

is nonempty. Consider $x \varepsilon B_R(M)$. By assumption, if $x_i = 0$ then $x \varepsilon F_i$. Moreover for any j,k such that $x_j \neq 0$, $x_k \neq 0$ it must be the case that $d(x,F_j) = d(x,F_k)$.

But $\bigcup_{i \varepsilon M} F_i = \Delta$, and so, for some $k \varepsilon M$, $d(x,F_k) = 0$. Since each set F_i is closed, $x \varepsilon F_i$, for each $i \varepsilon M$. Thus $\bigcap_{i \varepsilon M} F_i \neq \Phi$.

Q.E.D.

Theorem 3

Let v be a transferable value game, and M any coalition. Then $B_*(M)$, $B_1(M)$ and $K(M)$ are nonempty.

Proof: (i) If $v(M) = 0$ then, as in the proof of lemma 4, $x \varepsilon B_*(M)$ iff $x_i = 0$ for each $i \varepsilon M$.

Assume therefore that $v(M) \neq 0$. In this case the set $V(M)$ of payoffs is isomorphic to the $(m - 1)$-dimensional simplex Δ under the transformation:

$$\eta : V(M) \to \Delta : (x_1,\ldots,x_n) \to \frac{1}{v(M)} (x_i : i \varepsilon M).$$

We may therefore identify $V(M)$ with Δ. By lemma 5, the relation $P_*(x)$ is acyclic on M x M, for each $x \varepsilon \Delta$. By lemma 6, $\bigcup_{\varepsilon M} E_{P_*}(i) = \Delta$. But each $E_*(i)$ is the closure of $E_{P_*}(i)$, and so

$$\bigcup_{i \varepsilon M} E_*(i) = \Delta.$$

By definition 4, if $x \varepsilon \Delta$ such that $x_i = 0$ then $jP_*(x)i$ for no $j \varepsilon M$. Thus $\sum_i \subset E_*(i)$. Thus the family $\{E_*(i) : I \varepsilon M\}$ satisfies the conditions of Theorem 2, and so

$$B_*(M) = \bigcap_{i \varepsilon M} E_*(i) \text{ is nonempty.}$$

(ii) The proof for the kernel proceeds in the same way. From the definitions, the set $\Delta_{ji} = \{x \varepsilon \Delta : jQ(x)i\}$, for $i,j \varepsilon M\}$, is an open set in Δ. To see this note that $x \varepsilon \Delta_{ji}$ iff $S_{ji}(x) > S_{ij}(x)$ and

$x_i > 0$. But if

$$e_x(C) = v(C) - x(C)$$
$$> e_x(D) = v(D) - x(D)$$

for some $C \varepsilon T_{ji}$ and all $D \varepsilon T_{ij}$, then there exists a neighborhood U of x in Δ such that $S_{ji}(x') > S_{ij}(x')$ for all $x' \varepsilon U$. Therefore the i^{th} equilibrium set

$$E_Q(i) = \bigcap_{j \varepsilon M} (\Delta \backslash \Delta_{ji}) \text{ is closed.}$$

Moreover, if $x_i = 0$ then $jQ(x)i$ for no $j \varepsilon M$. Thus $\sum_i \subset E_Q(i)$. Again Theorem 2 shows that

$$B_Q(M) = \bigcap_{i \varepsilon M} E_Q(i) \text{ is nonempty}$$

Since $x \varepsilon B_Q(M)$ iff $iQ(x)j$ for no $i,j \varepsilon M$ clearly $B_Q(M) = K(M)$ and so the kernel of M is nonempty.

(iii) Since $B_*(M) \subset B_1(M)$ and $K(M) \subset B_1(M)$, proof that $B_*(M)$ or $K(M)$ is nonempty immediately gives that $B_1(M)$ is nonempty.

The result for $B_1(M)$ was obtained by Peleg (1967b), using acyclicity of the power relation P, together with a fixed point argument. The proof of Theorem 1 based on the KKM Theorem which has been presented here is due to Border.

Since the KKM Theorem can be extended (Fan, 1961) to the case where the underlying space is a general compact convex set, the proof procedure presented here can be extended to games without side payments (see Peleg, 1969; and Billera, 1970). Indeed the KKM-Fan theorem can be further extended (Schofield, 1984) to give a local result requiring no convexity assumptions. Moreover, Chichilinsky (1981) has recently obtained a purely topological extension of the KKM Theorem. Suppose that $\mathbb{F} = \{F_1,\ldots,F_m\}$ is a family of (contractible) subsets of R^m with the following properties: (i) if a subfamily has nonempty intersection then it is contractible; (ii) the union of any subfamily with at most (m + 1) members is contractible. Then $\mathbb{F}$ itself has a nonempty intersection if and only if the union of the family is contractible.

These extensions suggest that bargaining notions may be developed for spatial games that rely on the contractibility of the characteristic sets, rather than on their convexity properties.

5. A FINAL EXAMPLE

To illustrate the differences between the equilibrium sets $B_2(M)$, $B_*(M)$ and $B_1(M)$, consider example 1 again:

$$\left[\begin{array}{cccccccc} & 1 & 3 & 6 & & 2 & 4 & 5 \\ 8: & 1 & 2 & 5 & ; & 1 & 3 & 3 \end{array}\right]$$

First of all we compute $B_2(\{1,3,6\})$, supposing for convenience that $v(M) = 1$ for each winning coalition. Consider $x \varepsilon V(M)$ such that $x_6 < 1/2$. {6} may object to {1,3} forming $(y,\{4,5,6\})$ with $y_6 = 1/2$, $y_4 + y_5 = 1/2$. {1,3} need both {4,5} to counter object. Since $x_1 + x_3 > 1/2$, there is no counter. Thus $x \varepsilon B_2(\{1,3,6\})$ implies $x_6 \geq 1/2$. In the same way if $x_2 < 1/3$, then {3} objects to {1,6}

forming $(y,\{3,4,5\})$ with $y_3 = y_4 = y_5 = 1/3$. But {1,6} need either {4} or {5}. Since $x_1 + x_6 > 2/3$ it is impossible for them to counter object with either {4} or {5}. Thus $x \varepsilon B_2(\{1,3,6\})$ implies $x_3 \geq 1/3$. Finally if $x_1 < 1/4$, then {1} objects to {3,6} forming $(y,\{1,2,4,5\})$ with $y_1 = y_2 = y_4 = y_5 = 1/4$. To counter object {3,6} needs either {2}, {4} or {5}. Since $x_3 + x_6 > 3/4$ they have no counter objection. Thus

$$x \varepsilon B_2(\{1,3,6\}) \text{ implies } x_6 \geq 1/2,\ x_3 \geq 1/3, x_1 \geq 1/4.$$

However for $x \varepsilon V(M)$, $x_1 + x_3 + x_6 = 1$. Thus $B_2(\{1,3,6\}) = \Phi$. To compute $B_*(\{1,3,6\})$ we proceed as follows. Consider player {6} first of all. As we have seen if $x_6 \geq 1/2$ then it is not the case that {6} P(x) {1,3}. Suppose, however, that $x_6 < 1/2$. Then {6} P(x) {1,3}. Now {1} may block this objection if it can find a justified objection against {3,6}, and this it may do if $x_1 < 1/4$. In the same way {3} may block this objection if it can find a justified objection against {1,6}, and again this is possible if $x_3 < 1/3$. Thus

$$\begin{aligned} F_{P_*}(6) &= \{x \varepsilon \Delta : 6P_*(x)j \text{ for no } j = 1,3\} \\ &= \{x \varepsilon \Delta : x_6 \geq 1/2\} \cup \\ &\quad \{x \varepsilon \Delta : x_6 < 1/2,\ x_3 < 1/3,\ x_1 < 1/4\}. \end{aligned}$$

It should be clear that $B_*(\{1,3,6\}) = F_*(1) \cap F_*(3) \cap F_*(6)$ where $F_*(i)$ is the closure of $F_{P_*}(i)$ in V(M). Thus

$$B_*(\{1,3,6\}) = \{x \varepsilon \Delta : x_1 \leq 1/4,\ x_3 \leq 1/3,\ x_6 \leq 1/2\}.$$

As mentioned in Example 1, the "actual" payoff was

$$y = (0.19, 0.25, 0.56).$$

The square error of y from B_* is $\varepsilon = .085$. As noted in Example 1, the kernel for this coalition allocates 1/3 each to 1,3 and 6. The square error of the kernel is 0.26. Clearly the B_* prediction is superior to the kernel prediction in this example. More generally a recent empirical analysis (Schofield and Laver, 1983) of portfolio distribution in European government cabinets has found that the predictions made by B_2 (when it exists), or B_* otherwise, are superior to those of the kernel or the proportional payoffs (suggested by Gamson, 1961) in a significant number of cases.

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